An Osserman-type condition on $g.f.f$-manifolds with Lorentz metric*

Letizia Brunetti

Abstract

A condition of Osserman type, called φ-null Osserman condition, is introduced and studied in the context of Lorentz globally framed f-manifolds. An explicit example shows the naturalness of this condition in the setting of Lorentz $\mathcal{S}$-manifolds. We prove that a Lorentz $\mathcal{S}$-manifold with constant φ-sectional curvature is φ-null Osserman, extending a result stated for Lorentz Sasaki space forms. Then we state some characterizations for a particular class of φ-null Osserman $\mathcal{S}$-manifolds. Finally, some examples are examined.

2000 Mathematics Subject Classification. 53C50, 53C25, 53B30.
Keywords and phrases. Lorentz metric, Osserman condition, $g.f.f$-structure.

1 Introduction

The study of the behaviour of the Jacobi operators is an important topic in Riemannian and, more generally, in semi-Riemannian geometry. More precisely, let (M, g) be a Riemannian manifold with curvature tensor R and consider a point p in M. For any unit vector $X \in T_pM$, the symmetric endomorphism $R_X = R_p(\cdot, X)X : X^\perp \to X^\perp$ is called the Jacobi operator with respect to X. If the eigenvalues of R_X are independent of the choices of X and p, one says that (M, g) is an Osserman manifold ([17]).

Several results have been obtained looking for the solution of the Osserman conjecture, which states that an Osserman manifold is flat or it is locally a rank-one symmetric space ([7, 8, 9]). Osserman manifolds have been studied in the Lorentzian context ([4, 11, 12]), where a complete solution for the Osserman conjecture has been found. Recently, in [1], Atindogbe and Duggal have introduced and studied suitable operators of Jacobi type associated with a semi-Riemannian degenerate metric.

In ([12]) the authors defined the Jacobi operator $\bar{R}_u$, u being a null (or lightlike) vector tangent to a Lorentz manifold M. Given a unit timelike vector z tangent to M, they introduced and investigated the so-called null Osserman condition with respect to z (see also [13]).

Obviously, Lorentz almost contact manifolds are studied in this context. In particular, a Lorentz Sasaki space form, whose characteristic vector field ξ is timelike, is globally null Osserman with respect to ξ ([13]).
This result does not hold in the context of Lorentz globally framed f-manifolds $(M^{2n+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$, $s \geq 2$, as we will see with a counterexample.

This motivates the introduction of a more general condition of Osserman type, which we will call *φ-null Osserman condition.*

*The author wishes to express her thanks to professors A.M. Pastore and M. Falcitelli for helpful comments and for many stimulating conversations. The work was supported by the Research Program n. 01.08 of University of Bari

The main results of this paper state the links between the φ-null Osserman condition and the behaviour of the φ-sectional curvature in Lorentz $\mathcal{S}$-manifolds. After a preliminary section, where we gather some facts about $g.f.f$-manifolds, needed in the rest of the paper, in Section 3 we discuss the relationship between the null Osserman condition and the Lorentz $\mathcal{S}$-structures, giving an example of Lorentz $\mathcal{S}$-space form which does not satisfy the null Osserman conditions. We endow the compact Lie group $U(2)$ with a Lorentz $\mathcal{S}$-structure of rank 2. This manifold is an $\mathcal{S}$-space form with two characteristic vector fields ξ_1 and ξ_2, ξ_1 timelike, that does not satisfy the null Osserman condition with respect to ξ_1.

In Section 4 we introduce the notion of φ-null Osserman manifold, and we state that a Lorentz $\mathcal{S}$-manifold with constant φ-sectional curvature is φ-null Osserman with respect to the timelike characteristic vector field. We prove, in Section 5, the converse of such result for a particular class of φ-null Osserman Lorentz $\mathcal{S}$-manifolds. Namely, under some conditions on the eigenvalues of the Jacobi operator, we obtain that any φ-null Osserman manifold is a Lorentz $\mathcal{S}$-space form.

In particular, it is interesting to note that the existence of the only eigenvalue 1 of the Jacobi operator is related to the φ-sectional flatness of the manifold.

Finally in the case of 4-dimensional φ-null Osserman manifolds we find a compact example, using the Lie group $U(2)$, and also a non compact example.

All manifolds, tensor fields and maps are assumed to be smooth, moreover we suppose all manifolds are connected. We will use the Einstein convention omitting the sum symbol for repeated indexes. Following the notations of S. Kobayashi and K. Nomizu, for the curvature tensor R we have $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, and $R(X,Y,Z,W) = g(R(Z,W)Y,X)$, for any $X,Y,Z,W \in \mathfrak{X}(M)$. The *sectional curvature* $K_p(\pi)$ at p of a non-degenerate 2-plane $\pi = span\{X,Y\}$ is given by

$$K_p(\pi) = K_p(X,Y) = \frac{R_p(X,Y,X,Y)}{\Delta(\pi)} = \frac{g_p(R_p(X,Y)Y,X)}{\Delta(\pi)},$$

where $\Delta(\pi) = g(X,X)g(Y,Y) - g(X,Y)^2 \neq 0$.

2 Preliminaries

Following [3, 5, 19], we recall some definitions. An almost contact manifold is a $(2n+1)$-dimensional manifold M endowed with an almost contact structure, i.e. M^{2n+1} has a $(1,1)$-tensor field f such that $rank(f) = 2n$, a 1-form η and a vector field ξ satisfying $f^2(X) = -X + \eta(X)\xi$ and $\eta(\xi) = 1$. Moreover, if g is a semi-Riemannian metric on M^{2n+1} such that, for any $X,Y \in \mathfrak{X}(M^{2n+1})$,

$$g(fX,fY) = g(X,Y) - \varepsilon\eta(X)\eta(Y),$$

where $\varepsilon = \pm 1$ according to the causal character of ξ, M^{2n+1} is called an indefinite almost contact manifold. Such a manifold is said to be an indefinite contact manifold if $d\eta = \Phi$, Φ being defined by $\Phi(X,Y) = g(X,fY)$. Furthermore, if the structure (f,ξ,η) is normal, i.e. $N = [f,f] + 2d\eta\otimes\xi = 0$, then the indefinite contact structure is called an indefinite Sasaki structure and, in this case, the manifold $(M^{2n+1}, f, \xi, \eta, g)$ is called indefinite Sasaki.

In the Riemannian case a generalization of these structures have been studied by Blair in [3], by Goldberg and Yano in [15]. In [5] we studied such structures in semi-Riemannian context.

A manifold M is called a globally framed f-manifold (briefly $g.f.f$-manifold) if it is endowed with a non null $(1,1)$-tensor field φ of constant rank, such that $\ker\varphi$ is parallelizable i.e. there exist global vector fields ξ_α, $\alpha \in \{1,\dots,s\}$, and 1-forms η^α, satisfying

$$\varphi^2 = -I + \eta^\alpha \otimes \xi_\alpha \text{ and } \eta^\alpha(\xi_\beta) = \delta^\alpha_\beta.$$

A $g.f.f$-manifold $(M^{2n+s}, \varphi, \xi_\alpha, \eta^\alpha)$, $\alpha \in \{1, \ldots, s\}$, is said to be an indefinite $g.f.f$-manifold if g is a semi-Riemannian metric satisfying the following compatibility condition

$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon_\alpha \eta^\alpha(X)\eta^\alpha(Y)$$

for any vector fields X, Y, being $\varepsilon_\alpha = \pm 1$ according to whether ξ_α is spacelike or timelike. Then, for any $\alpha \in \{1, \ldots, s\}$ and $X \in \mathfrak{X}(M^{2n+s})$, one has $\eta^\alpha(X) = \varepsilon_\alpha g(X, \xi_\alpha)$.

An indefinite $g.f.f$-manifold is an indefinite $\mathcal{S}$-manifold if it is normal and $d\eta^\alpha = \Phi$, for any $\alpha \in \{1, \ldots, s\}$, where $\Phi(X, Y) = g(X, \varphi Y)$ for any $X, Y \in \mathfrak{X}(M^{2n+s})$. The normality condition is expressed by the vanishing of the tensor field $N = N_\varphi + 2d\eta^\alpha \otimes \xi_\alpha$, N_φ being the Nijenhuis torsion of φ.

Furthermore, as proved in [5], the Levi-Civita connection of an indefinite $\mathcal{S}$-manifold satisfies:

$$(\nabla_X \varphi)Y = g(\varphi X, \varphi Y)\widetilde{\xi} + \widetilde{\eta}(Y)\varphi^2(X),$$

where $\widetilde{\xi} = \sum_{\alpha=1}^{s} \xi_\alpha$ and $\widetilde{\eta} = \varepsilon_\alpha \eta^\alpha$. Note that, for $s = 1$, we reobtain the notion of indefinite Sasaki manifold.

We recall that $\nabla_X \xi_\alpha = -\varepsilon_\alpha \varphi X$ and $\ker \varphi$ is an integrable flat distribution since $\nabla_{\xi_\alpha} \xi_\beta = 0$, for any $\alpha, \beta \in \{1, \ldots, s\}$. Anyway, an indefinite $\mathcal{S}$-manifold is never flat since $K(X, \xi_\alpha) = \varepsilon_\alpha$ for any non lightlike $X \in \operatorname{Im} \varphi_p$.

For more details we refer to [5], where we describe three examples of non compact indefinite $\mathcal{S}$-manifolds. More precisely we construct two different indefinite $\mathcal{S}$-structures with metrics of index $\nu = 2$ on $\mathbb{R}^6$ and an indefinite $\mathcal{S}$-structure with Lorentz metric on $\mathbb{R}^4$. Moreover, in [6] we give explicit examples of compact indefinite $g.f.f$-manifolds and indefinite $\mathcal{S}$-manifolds.

We also remark that every $g.f.f$-manifold is subject to the following topological condition: it has to be either non compact or compact with vanishing Euler characteristic, since it admits never vanishing vector fields. This implies that such a $g.f.f$-manifold always admits Lorentz metrics.

Let us fix few notation about curvature tensor field. As usual, a 2-plane $\pi = span\{X, \varphi X\}$ in T_pM, with $p \in M$ and $X \in \operatorname{Im} \varphi_p$, is said to be a *$\varphi$-plane* and the sectional curvature at p of such a plane, with X a non lightlike vector, is called the *φ-sectional curvature* at p and is denoted by $H_p(X)$.

An indefinite $\mathcal{S}$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ is said to be an indefinite $\mathcal{S}$-space form if the φ-sectional curvature $H_p(X)$ is constant, for any point and any φ-plane. In particular, in [5] it is proved that an indefinite $\mathcal{S}$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ is an indefinite $\mathcal{S}$-space form with $H_p(X) = c$ if and only if the Riemannian $(0, 4)$-type curvature tensor field R is given by

$$\begin{aligned} R(X, Y, Z, W) = &-\frac{c + 3\varepsilon}{4}\{g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)\} \\ &- \frac{c - \varepsilon}{4}\{\Phi(W, X)\Phi(Z, Y) - \Phi(Z, X)\Phi(W, Y) \\ &+ 2\Phi(X, Y)\Phi(W, Z)\} - \{\widetilde{\eta}(W)\widetilde{\eta}(X)g(\varphi Z, \varphi Y) \\ &- \widetilde{\eta}(W)\widetilde{\eta}(Y)g(\varphi Z, \varphi X) + \widetilde{\eta}(Y)\widetilde{\eta}(Z)g(\varphi W, \varphi X) \\ &- \widetilde{\eta}(Z)\widetilde{\eta}(X)g(\varphi W, \varphi Y)\}, \end{aligned} \tag{1}$$

for any vector fields X, Y, Z and W on M, where $\varepsilon = \sum_{\alpha=1}^{s} \varepsilon_\alpha$.

In regard to the curvature tensor of an indefinite $\mathcal{S}$-manifold, it is important to recall the

following formulas, for any $X, Y, Z, W \in \operatorname{Im}\varphi$ and any $\alpha, \beta, \gamma, \delta \in \{1, \dots, s\}$:

$$\begin{aligned}
&R(X, \xi_\alpha, X, Y) = \varepsilon_\alpha g(X, X) g(\widetilde{\xi}, Y) = 0, \\
&R(\xi_\alpha, X, \xi_\beta, Y) = \varepsilon_\alpha \varepsilon_\beta g(X, Y), \\
&R(\xi_\alpha, X, \xi_\beta, \xi_\gamma) = \varepsilon_\alpha \varepsilon_\beta g(X, \xi_\gamma) = 0, \\
&R(\xi_\alpha, \xi_\delta, \xi_\beta, \xi_\gamma) = 0, \\
&R(X, Y, \varphi Z, W) + R(X, Y, Z, \varphi W) = \varepsilon P(X, Y; Z, W),
\end{aligned} \tag{2}$$

where $P(X, Y; Z, W) = \Phi(X, Z) g(Y, W) - \Phi(X, W) g(Y, Z) - \Phi(Y, Z) g(X, W) + \Phi(Y, W) g(X, Z)$.

Finally, we recall some useful properties for a curvature-like algebraic tensor. Let (V, g) be a pseudo-Euclidean real vector space of index ν, $0 < \nu < \dim V$. A multilinear map $F : V^4 \to \mathbb{R}$ is called a curvature-like map (or curvature-like algebraic tensor) if it satisfies the following conditions

$$\begin{aligned}
&F(y, x, z, w) = -F(x, y, z, w), \\
&F(z, w, x, y) = F(x, y, z, w), \\
&F(x, y, z, w) + F(x, z, w, y) + F(x, w, y, z) = 0.
\end{aligned}$$

For any non-degenerate 2-plane $\pi = span\{z, w\}$ in V it is possible to define the number

$$k(z, w) = \frac{F(z, w, z, w)}{\Delta(\pi)}.$$

If $k(z, w)$ is constant for any non-degenerate 2-plane and $k(z, w) = k$ then one gets $F(x, y, z, w) = k\,(g(x, z) g(y, w) - g(y, z) g(x, w))$. Now, arguments similar to those in Proposition 28 ([16, page 229]), can be used to prove the following result.

Lemma 2.1. *Let (V, g) be a Lorentz real vector space and $F : V^4 \to \mathbb{R}$ a curvature-like map. Then the following conditions are equivalent.*

a) $F(x, y, z, w) = k\,(g(x, z) g(y, w) - g(y, z) g(x, w))$,

b) $F(x, y, y, x) = 0$ *for any degenerate plane* $\pi = span\{x, y\}$ *in* V.

3 Null Osserman condition and Lorentz $\mathcal{S}$-manifolds

It is well-known that a Lorentz manifold has constant sectional curvature at a point p if and only if it satisfies the Osserman condition at p.

Contrary to this, no Lorentz $\mathcal{S}$-manifold can satisfy the Osserman condition since, as remarked in Section 2, a Lorentz $\mathcal{S}$-manifold can not have constant sectional curvature.

In [12] the authors introduce another Osserman condition, named the null Osserman condition. Namely, let (M, g) be a Lorentz manifold, $p \in M$ and u a null vector in T_pM. Then the orthogonal complement $u^\perp$ of u is a degenerate vector space since $span\{u\} \subset u^\perp$. So one considers the quotient space $\bar{u}^\perp = u^\perp / span\{u\}$ and the canonical projection $\pi : u^\perp \to \bar{u}^\perp$. It is possible to define a positive definite inner product $\bar{g}$ on $\bar{u}^\perp$ putting

$$\bar{g}(\bar{x}, \bar{y}) = g(x, y),$$

where, for any $x, y \in u^\perp$, $\bar{x} = \pi(x)$ and $\bar{y} = \pi(y)$.

From now on, every bar-object will stand for geometrical objects related to $\bar{u}^\perp$. So, fixed a null vector $u \in T_pM$, the Jacobi operator with respect to u can be defined by the linear map $\bar{R}_u : \bar{u}^\perp \to \bar{u}^\perp$ such that $\bar{R}_u \bar{x} = \pi(R(x, u)u)$ ([12] and Definition 3.2.1 in [13]).

Clearly, $\bar{R}_u$ is self-adjoint with respect to $\bar{g}$, hence $\bar{R}_u$ is diagonalizable.

In Lorentzian geometry it is well-known that a null vector u and a timelike vector z are never orthogonal. Hence, in a Lorentz manifold (M, g), the null congruence set determined by a timelike vector $z \in T_pM$ at p, denoted by $N(z)$, is defined by

$$N(z) = \{u \in T_pM \mid g(u, u) = 0,\ g(u, z) = -1\}.$$

A Lorentz manifold (M, g) is called null Osserman with respect to a unit timelike vector $z \in T_pM$ at a point p if the characteristic polynomial of $\bar{R}_u$ is independent of $u \in N(z)$. Let L be a timelike line subbundle of TM. If (M, g) is null Osserman with respect to each unit timelike vector $z \in L$, then (M, g) is called pointwise null Osserman with respect to L. Moreover, if (M, g) is pointwise null Osserman with respect to L and the characteristic polynomial of $\bar{R}_u$ is independent of the choice of a unit $z \in L$, then (M, g) is said to be globally null Osserman with respect to L.

Another set associated to a unit timelike vector z in T_pM is the celestial sphere $S(z)$ of z given by

$$S(z) = \{x \in z^{\perp} \mid g(x, x) = 1\}.$$

According to a result in [13], using the celestial sphere of z, one can obtain all the elements of $N(z)$. In fact one has

$$\forall\, u \in N(z)\ \exists|\, x \in S(z) \text{ such that } u = z + x.$$

It is very natural to use this definition in the context of Lorentz contact manifolds. In particular, as stated in [13], Lorentz Sasaki space forms are globally null Osserman with respect to the timelike characteristic vector field. An easy example shows that in a Lorentz $\mathcal{S}$-space form the null Osserman condition with respect to a timelike characteristic vector does not hold.

Indeed, considering the 4-dimensional manifold $U(2)$ and the Lie algebra $\mathfrak{u}(2)$, we denote by ξ_1, ξ_2, X, Y the left-invariant vector fields on $U(2)$, determined, in the same order, by the basis $\{\imath E_{11}, -\imath E_{22}, E_{12} - E_{21}, \imath(E_{12} + E_{21})\}$ of $\mathfrak{u}(2)$, where $(E_{ij})_{i,j\in\{1,2\}}$ is the canonical basis of $gl(2, \mathbb{C})$. Then, we get:

$$[X, Y] = 2\xi_1 + 2\xi_2, \quad [X, \xi_\alpha] = -Y, \quad [Y, \xi_\alpha] = X, \quad [\xi_\alpha, \xi_\beta] = 0$$

for any $\alpha, \beta \in \{1, 2\}$. Let us consider the left-invariant 1-forms η^1 and η^2 determined by the dual 1-forms of $\imath E_{11}$ and $-\imath E_{22}$, respectively, and the left-invariant tensor field φ such that $\varphi(X) = Y$, $\varphi(Y) = -X$ and $\varphi(\xi_1) = \varphi(\xi_2) = 0$. The manifold $U(2)$ is compact, connected, with Euler number $\chi(U(2)) = 0$, thus we can define a left-invariant Lorentz metric g such that the vector fields ξ_1, ξ_2, X, Y form an orthonormal basis with $g(\xi_1, \xi_1) = -1$.

Hence, we obtain a normal indefinite $g.f.f$-structure and its associated Sasaki 2-form Φ verifies $\Phi = d\eta^\alpha$, for any $\alpha \in \{1, 2\}$, so that it turns out to be a Lorentz $\mathcal{S}$-structure on $U(2)$. Moreover, one sees at once that $U(2)$ has constant φ-sectional curvature 4 ([6]). We see that $U(2)$ does not verify the null Osserman condition with respect to ξ_1. In fact, putting

$$u_1 = X + \xi_1,\ u_2 = Y + \xi_1,\ u_3 = \xi_2 + \xi_1,$$

one has $u_1, u_2, u_3 \in N(\xi_1)$. By (1), we have

$$R(Y, u_1)u_1 = Y + 3g(Y, \varphi u_1)\varphi u_1 + \widetilde{\eta}(u_1)\widetilde{\eta}(u_1)Y = 5Y,$$

$$R(\xi_2, u_1)u_1 = \sum_{\alpha=1}^{2} \xi_\alpha + X = \xi_2 + u_1.$$

Analogously, for u_2, we obtain

$$R(X, u_2)u_2 = X + 3X + X = 5X,$$
$$R(\xi_2, u_2)u_2 = \sum_{\alpha=1}^{2} \xi_\alpha + Y = \xi_2 + u_2.$$

For any $z \in u_3^\perp$, we have

$$R(z, u_3)u_3 = -\widetilde{\eta}(u_3)\widetilde{\eta}(u_3)\varphi^2 z = 0,$$

since $\widetilde{\eta}(u_3) = 0$.

Then it is evident that the eigenvalues of $\bar{R}_{u_1}$ and $\bar{R}_{u_2}$ are 5 and 1 whereas $\bar{R}_{u_3} = 0$.

4 φ-Null Osserman Condition

In this section, inspired by the example of $U(2)$, we introduce a new Osserman condition that will be applied to Lorentz $g.f.f$-manifolds.

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, \ldots, s\}$, be a Lorentz $g.f.f$-manifold, it is easy to check that the timelike vector field must be a characteristic vector field. Without loss of generality we can assume that ξ_1 is the timelike vector field.

Taking in mind the example in Section 3, we claim that, if $s \geq 2$, then the flatness of $\ker\varphi$ influences the behaviour of the Jacobi operators $\bar{R}_{u_\alpha}$ with $u_\alpha = \xi_1 + \xi_\alpha$, for any $\alpha \in \{2, \ldots, s\}$. Since the matter is related to the null vector u_α, we give the following Osserman condition.

Given a point p of M, the set

$$S_\varphi((\xi_1)_p) = S((\xi_1)_p) \cap \operatorname{Im}\varphi_p,$$

is called the *φ-celestial sphere* of $(\xi_1)_p$ at p. We define the analogous of the null congruence set, called the φ-null congruence set, denoted by $N_\varphi((\xi_1)_p)$, putting

$$N_\varphi((\xi_1)_p) = \{u \in T_pM \mid u = (\xi_1)_p + x,\ x \in S_\varphi((\xi_1)_p)\}.$$

Since the Osserman conditions are formulated pointwise, to simplify the notation we omit any reference to the point, there is no ambiguity.

Now, we are ready to state the definition of φ-null Osserman condition with respect to the timelike vector ξ_1.

Definition 4.1. Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentz $g.f.f$-manifold, $\dim M = 2n + s$, $n, s \geq 1$, with timelike vector field ξ_1 and consider $p \in M$. M is called φ-null Osserman with respect to ξ_1 if the characteristic polynomial of $\bar{R}_u$ is independent of $u \in N_\varphi(\xi_1)$, that is the eigenvalues of $\bar{R}_u$ are independent of $u \in N_\varphi(\xi_1)$.

Remark 4.2. If $(M, \varphi, \xi, \eta, g)$ is a Lorentz almost contact manifold, then it can be considered as a Lorentz $g.f.f$-manifold with $\alpha = 1$. Obviously one has $S(\xi) = S_\varphi(\xi)$ and $N(\xi) = N_\varphi(\xi)$. It follows that the null Osserman condition with respect to ξ coincides with the φ-null Osserman condition.

It is clear that $U(2)$ verifies the φ-null Osserman condition with respect to ξ_1. In fact, we consider an arbitrary unit vector z of $\operatorname{Im}\varphi_p$ putting $z = aX + bY$. Putting $u_4 = z + \xi_1$, we have $u_4 \in N(\xi_1)$ and

$$u_4^\perp = span\{X + a\xi_1, Y + b\xi_1, \xi_2\} = span\{\varphi u_4, u_4, \xi_2\}.$$

Then, we get

$$R(\varphi u_4, u_4)u_4 = \varphi u_4 + 3\varphi u_4 + \varphi u_4 = 5\varphi u_4, \tag{3}$$

$$R(\xi_2, u_4)u_4 = \sum_{\alpha=1}^{2} \xi_\alpha - \varphi^2 u_4 = \xi_2 + \xi_1 + z = \xi_2 + u_4.$$

It follows that, for any $u = z + \xi_1$ in $N(\xi_1)$ with $z \in \operatorname{Im}\varphi_p$ and $g(z, z) = 1$, the eigenvalues of $\bar{R}_u$ are 5 and 1.

Taking into account the analogous definitions in [13], we introduce the concept of φ-null Osserman condition with respect to a timelike line bundle.

Definition 4.3. Let L be the timelike line subbundle spanned by ξ_1. If (M, g) is φ-null Osserman with respect to ξ_1 and $-\xi_1$, then (M, g) is called pointwise φ-null Osserman with respect to L. Moreover, if (M, g) is pointwise φ-null Osserman with respect to L and the characteristic polynomial of $\bar{R}_u$ is independent of the choice of a unit $z \in L$, then (M, g) is said to be globally φ-null Osserman with respect to L.

Again we consider $U(2)$ and the null vector $u_5 = z - \xi_1$, with z arbitrary unit vector in $\operatorname{Im}\varphi_p$ as before. For any $x \in u_5^\perp$ one gets

$$\begin{aligned} R(x, u_5)u_5 &= -2\varphi^2 x + g(\varphi x, \varphi u_5)\varphi^2 u_5 + 3g(x, \varphi u_5)\varphi u_5 \\ &\quad + \widetilde{\eta}(x)\sum_{\alpha=1}^{2} \xi_\alpha - g(\varphi x, \varphi u_5)\sum_{\alpha=1}^{2} \xi_\alpha + \widetilde{\eta}(x)\varphi^2 u_5. \end{aligned}$$

Then it is clear that

$$R(\varphi u_5, u_5)u_5 = 5\varphi u_5, \quad R(\xi_2, u_5)u_5 = \xi_2 - u_5. \tag{4}$$

It follows that $U(2)$ is φ-null Osserman with respect to $-\xi_1$, too. The formulas (3) and (4) show that the characteristic polynomial is the same, so $U(2)$ is globally φ-null Osserman with respect to the timelike line subbundle $L = \langle \xi_1 \rangle$.

In the next theorem we prove, more generally, that each Lorentz $\mathcal{S}$-space form satisfies the φ-null Osserman condition.

Theorem 4.4. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\dim M = 2n + s$, be a Lorentz $\mathcal{S}$-manifold with ξ_1 timelike and constant φ-sectional curvature. Then M verifies the φ-null Osserman condition with respect to the timelike characteristic vector field.*

Proof. Denoting by c the φ-sectional curvature, (1) holds with $\varepsilon = s - 2$.

Let u be a vector in $N_\varphi(\xi_1)$, hence $u = \xi_1 + x_1$ with $x_1 \in S_\varphi(\xi_1)$, and consider $x \in u^\perp$. We have:

$$g(\varphi u, \varphi u) = g(u, u) - \sum_{\alpha=1}^{s} \varepsilon_\alpha \eta^\alpha(u)\eta^\alpha(u) = \eta^1(u)\eta^1(u) = 1, \tag{5}$$

$$g(\varphi x, \varphi u) = g(x, u) - \sum_{\alpha=1}^{s} \varepsilon_\alpha \eta^\alpha(x)\eta^\alpha(u) = \eta^1(x). \tag{6}$$

By (1), (5) and (6) we compute $R(x, u, u, w)$ for any $w \in T_pM$, obtaining

$$\begin{aligned} R(x, u, u, w) = &-\frac{c + 3(s-2)}{4}\{g(\varphi x, \varphi w) - \eta^1(x)g(\varphi u, \varphi w)\} \\ &- \frac{3}{4}(c - s + 2)g(x, \varphi u)g(w, \varphi u) \\ &- \{\widetilde{\eta}(w)\widetilde{\eta}(x) + \widetilde{\eta}(w)\eta^1(x) + g(\varphi w, \varphi x) + \widetilde{\eta}(x)g(\varphi w, \varphi u)\}. \end{aligned} \tag{7}$$

Now, being $\dim M = 2n+s$, we consider $\{x_1, \varphi x_1, x_3, \ldots, x_{2n}\}$ as an orthonormal base of $\operatorname{Im}\varphi_p$, which determines the bases $\mathfrak{B} = \{u, \varphi x_1, \xi_2, \ldots, \xi_s, x_3, \ldots, x_{2n}\}$ of $u^\perp$ and $\overline{\mathfrak{B}} = \{\overline{\varphi x}_1, \bar{\xi}_2, \ldots, \bar{\xi}_s, \bar{x}_3, \ldots, \bar{x}_{2n}\}$ of $\bar{u}^\perp$. For brevity, we also denote them by $\mathfrak{B} = \{e_i\}_{1\le i\le m}$, $\bar{\mathfrak{B}} = \{\bar{e}_i\}_{1\le i\le m-1}$, being $m = 2n+s-1$. In general, for any $x \in u^\perp$, one has

$$\bar{R}_u(\bar{x}) = -\sum_{i=1}^{m-1} R(x, u, u, e_i)\bar{e}_i. \tag{8}$$

By (7) and (8) we obtain

$$\begin{aligned}
\bar{R}_u(\overline{\varphi x}_1) &= \{\frac{c+3(s-2)}{4} + \frac{3}{4}(c-s+2)\}\overline{\varphi x}_1 + \overline{\varphi x}_1 = (c+1)\overline{\varphi x}_1,\\
\bar{R}_u(\bar{x}_j) &= \frac{c+3(s-2)}{4}\bar{x}_j + \bar{x}_j = \frac{c+3s-2}{4}\bar{x}_j, \quad \forall j \in \{2, \ldots, 2n\}\\
\bar{R}_u(\bar{\xi}_\beta) &= \sum_{\gamma=2}^{s} \widetilde{\eta}(\xi_\beta)\widetilde{\eta}(\xi_\gamma)\bar{\xi}_\gamma = \sum_{\gamma=2}^{s} \bar{\xi}_\gamma, \quad \forall \beta \in \{2, \ldots, s\}.
\end{aligned}$$

It follows that the representation matrix of $\bar{R}_u$ with respect to $\bar{\mathfrak{B}}$ is independent of the choice of $u \in N_\varphi(\xi_1)$. In particular, it is easy to compute that the other eigenvalues are 0 and $s-1$, having eigenvectors $\bar{x}_\alpha = \bar{\xi}_2 - \bar{\xi}_\alpha$, $\alpha \in \{3, \ldots, s\}$, and $\bar{x} = \sum_{\beta=2}^{s} \bar{\xi}_\beta$, respectively. This completes the proof. ☐

In the following remark we note that, as for $U(2)$, each Lorentz $\mathcal{S}$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\dim M = 2n+s$, with constant φ-sectional curvature is globally φ-null Osserman with respect to $L = \langle \xi_1 \rangle$.

Remark 4.5. Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\dim M = 2n+s$, be a Lorentz $\mathcal{S}$-manifold with constant φ-sectional curvature. Considering the unit timelike vector field $\xi_1^- = -\xi_1$ and $x_1 \in S_\varphi(\xi_1^-)$, the vector $u^- = x_1 + \xi_1^-$ is a null vector of $N_\varphi(\xi_1^-)$. An easy computation shows that

$$\begin{aligned}
\bar{R}_{u^-}\overline{\varphi x}_1 &= \{\frac{c+3(s-2)}{4} + \frac{3}{4}(c-s+2)\}\overline{\varphi x}_1 + \overline{\varphi x}_1 = (c+1)\overline{\varphi x}_1,\\
\bar{R}_{u^-}\bar{x}_j &= \frac{c+3(s-2)}{4}\bar{x}_j + \bar{x}_j = \frac{c+3s-2}{4}\bar{x}_j, \quad \forall j \in \{2, \ldots, 2n\}\\
\bar{R}_{u^-}\bar{\xi}_\beta &= \sum_{\gamma=2}^{s} \widetilde{\eta}(\xi_\beta)\widetilde{\eta}(\xi_\gamma)\bar{\xi}_\gamma = \sum_{\gamma=2}^{s} \bar{\xi}_\gamma, \quad \forall \beta \in \{2, \ldots, s\}.
\end{aligned}$$

Then M is φ-null Osserman with respect to ξ_1^-. Hence, considering the timelike line subbundle $L = \langle \xi_1 \rangle$, M is globally φ-null Osserman since it is φ-null Osserman with respect to each unit timelike vector $z \in L$ and the characteristic polynomial is independent of the choice of a unit timelike vector $z \in L$.

5 φ-null Osserman condition on Lorentz $\mathcal{S}$-manifolds

Following [12, 13], we recall that if (M, g) is a Lorentz manifold and u is a null vector of T_pM then a non-degenerate subspace $W \subset u^\perp$ such that $\dim W = \dim \bar{u}^\perp$ is called a *geometric realization* of $\bar{u}^\perp$. Let $\pi|_W : (W, g) \to (\bar{u}^\perp, g)$ be an isometry where, to simplify, we use the same letter g for non-degenerate metrics on W and $\bar{u}^\perp$. Moreover, a vector $x \in W$ is said to be a *geometrically*

realized eigenvector of $\bar{R}_u$ in W corresponding to an eigenvalue λ if $\pi|_W(x) = \bar{x}$ is an eigenvector of $\bar{R}_u$ with eigenvalue λ ([13]).

We are going to prove a theorem in order to describe the curvature tensor field of φ-null Osserman manifolds with two characteristic vector fields. An analogous statement can be found in different contexts ([13]). For this purpose we give some preliminary remarks about a null vector of a Lorentz $\mathcal{S}$-manifold with two characteristic vector fields and then we prove a lemma.

Remark 5.1. Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, 2\}$, be a Lorentz $\mathcal{S}$-manifold with timelike vector field ξ_1 and u a null vector in T_pM, $p \in M$. Since $TM = \operatorname{Im}\varphi \oplus \ker\varphi$, one can write u in the following way

$$u = \lambda x + a\,\xi_1 + b\,\xi_2,$$

where $x \in \operatorname{Im}\varphi$ with $g(x, x) = 1$. Being u a null vector, we have $\lambda^2 + b^2 = a^2$ therefore there exists $\theta \in [0, 2\pi[$ such that u can be written as follows

$$u = a(\cos\theta\, x + \xi_1 + \sin\theta\, \xi_2),$$

and it is not a restriction to use

$$u = \cos\theta\, x + \xi_1 + \sin\theta\, \xi_2, \tag{9}$$

For $\cos\theta \neq 0$ consider the vector $w = \tan\theta\, \xi_1 + \frac{1}{\cos\theta}\xi_2$. It is easy to check that w is a unit vector orthogonal to u, therefore

$$u^\perp = span\{u, \varphi x, x_2, \varphi x_2, \dots x_n, \varphi x_n, w\}.$$

Any $y \in u^\perp$ can be written as

$$y = \rho u + y' + \kappa w, \tag{10}$$

where $y' \in span\{\varphi x, x_2, \varphi x_2, \dots x_n, \varphi x_n\} \subset \operatorname{Im}\varphi_p$ and $\rho, \kappa \in \mathbb{R}$.

We need to define two $(1,3)$-type tensors S^* and S_* putting

$$\begin{aligned} S^*(x, y)v &= \widetilde{\eta}(y)\widetilde{\eta}(v)x - \widetilde{\eta}(x)\widetilde{\eta}(v)y + g(y, v)\widetilde{\eta}(x)\widetilde{\xi} - g(x, v)\widetilde{\eta}(y)\widetilde{\xi}, \\ S_*(x, y)v &= -g(\varphi y, \varphi v)\varphi^2 x + g(\varphi x, \varphi v)\varphi^2 y. \end{aligned}$$

Remark 5.2. If $u \in N_\varphi(\xi_1)$ and $y \in span\{\varphi x, x_2, \varphi x_2, \dots x_n, \varphi x_n\} \subset \operatorname{Im}\varphi$, then

$$g(S^*(u, y)u, y) - g(S_*(u, y)u, y) = 0.$$

The following lemma allows to state the expression of a curvature-like map F when F vanishes on a particular type of degenerate 2-plane and it has a suitable behaviour with respect to the characteristic vector fields.

Lemma 5.3. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, 2\}$, be a Lorentz g.f.f-manifold with timelike vector field ξ_1. Fixed a point $p \in M$, let $F : (T_pM)^4 \to \mathbb{R}$ be a curvature-like map such that, for any $x, y, v \in \operatorname{Im}\varphi$ and any $\alpha, \beta, \gamma \in \{1, 2\}$,*

$$F(x, \xi_\alpha, y, v) = 0,\ F(\xi_\alpha, x, \xi_\beta, y) = \varepsilon_\alpha\varepsilon_\beta g(x, y),\ F(\xi_\alpha, x, \xi_\beta, \xi_\gamma) = 0,\ F(\xi_1, \xi_2, \xi_1, \xi_2) = 0. \tag{11}$$

Then the following statements are equivalent.

a) F vanishes on any degenerate 2-plane $\pi = span\{u, y\}$, with $u \in N_\varphi(\xi_1)$ and $y \in u^\perp \cap \operatorname{Im}\varphi$,

b) $F(x, y, v, z) = g(S_*(x, y)v, z) - g(S^*(x, y)v, z)$.

Proof. An easy computation, using Remark 5.2, shows that *b*) ⇒ *a*).

Conversely, fix $p \in M$ and consider the curvature-like map H such that, for any $x, y, z, v \in T_pM$,

$$H(x, y, v, z) = F(x, y, z, w) - g(S_*(x, y)v, z) + g(S^*(x, y)v, z). \tag{12}$$

Condition *a*) implies that H vanishes on any degenerate 2-plane $span\{u, y\}$, for any $u \in N_\varphi(\xi_1)$ and $y \in u^\perp \cap \operatorname{Im}\varphi$, by Remark 5.2. We start proving that H vanishes on any degenerate 2-plane. To see this, let u be a null vector of T_pM as in (9) such that $\cos\theta \neq 0$. By the hypotheses and using (10), for any $y \in u^\perp$ we have

$$\begin{aligned}
g(S_*(u, y)u, y) &= (\rho g(\varphi u, \varphi u) + g(\varphi u, \varphi y'))^2 - g(\varphi u, \varphi u)\left(\rho^2 g(\varphi u, \varphi u) + g(\varphi y', \varphi y')\right)\\
&= \rho^2 g(\varphi u, \varphi u)^2 - \rho^2 g(\varphi u, \varphi u)^2 - g(\varphi y', \varphi y')g(\varphi u, \varphi u) = -g(y', y')g(\varphi u, \varphi u),\\
g(S^*(u, y)u, y) &= -\widetilde{\eta}(u)\widetilde{\eta}(u)g(y, y),\\
F(u, y, u, y) &= F(u, y', u, y') + 2\kappa F(u, y', u, w) + \kappa^2 F(u, w, u, w) = \cos^2\theta F(x, y', x, y')\\
&\quad + (1 - \sin\theta)^2(g(y', y') + \kappa^2) = g(\varphi u, \varphi u)F(x, y', x, y') + \widetilde{\eta}(u)\widetilde{\eta}(u)g(y, y)\\
&= g(\varphi u, \varphi u)F(u', y', u', y') - g(\varphi u, \varphi u)g(y', y') + \widetilde{\eta}(u)\widetilde{\eta}(u)g(y, y),
\end{aligned}$$

where $u' = x + \xi_1$ which belongs to $N_\varphi(\xi_1)$. Hence one obtains

$$H(u, y, u, y) = g(\varphi u, \varphi u)F(u', y', u', y'), \tag{13}$$

with $u' = x + \xi_1$ and $y \in u^\perp \cap \operatorname{Im}\varphi$.

If $\cos\theta = 0$, then $u = \xi_1 \pm \xi_2$ and $u^\perp = span\{u\} \oplus \operatorname{Im}\varphi$. By direct computation, it is easy to verify that

$$H(u, y, u, y) = 0, \tag{14}$$

for any $y \in u^\perp$.

Equations (13) and (14) clearly imply that H vanishes on any degenerate 2-plane. Applying Lemma 2.1 to H one has

$$F(x, y, v, z) = k\,(g(x, v)g(y, z) - g(y, v)g(x, z)) + g(S_*(x, y)v, z) - g(S^*(x, y)v, z). \tag{15}$$

By definition of k, using the hypotheses and (12), we deduce

$$k = \frac{H(\xi_\alpha, x, \xi_\alpha, x)}{\varepsilon_\alpha g(x, x)} = \frac{F(\xi_\alpha, x, \xi_\alpha, x) - g(x, x)}{\varepsilon_\alpha g(x, x)} = 0.$$

Then, substituting in (15), we obtain our assertion. □

Now, we define the following tensor fields of type $(1, 3)$, evaluating them at the point p:

$$R^0_I(x, y)v = g(y^I, v^I)x^I - g(x^I, v^I)y^I, \quad R^\varphi(x, y)v = g(\varphi y, v)\varphi x - g(\varphi x, v)\varphi y + 2g(x, \varphi y)\varphi v,$$

where x^I is the projection of x on $\operatorname{Im}\varphi$, for any tangent vector x.

It is useful to note that R^φ vanishes on the triplets containing a characteristic vector and that $R^0_I(x, y)v$ is orthogonal to ξ_1, for any $x, y, v \in T_pM$.

Now we are ready to prove the following result.

Theorem 5.4. *Let* $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, 2\}$ *and* $n > 1$, *be a* $(2n + 2)$*-dimensional Lorentz* $\mathcal{S}$*-manifold with timelike vector field* ξ_1*. The following three statements are equivalent.*

- *a*) 1. *M is* φ*-null Osserman with respect to* ξ_1 *and for any* $u \in N_\varphi(\xi_1)$ *the Jacobi operator* $\bar{R}_u$ *has eigenvalues* c_1 *and* c_2 *with multiplicities* 1 *and* $2(n - 1)$*, respectively.*

2. *Putting $u = \xi_1 + x_1$, with $x_1 \in S_\varphi(\xi_1)$, then φx_1 is a geometrically realized eigenvector of $\bar{R}_u$ in $u^\perp \cap \xi_1^\perp$ corresponding to the eigenvalue c_1, and any vector $y \in \operatorname{Im}\varphi \cap u^\perp$, orthogonal to φx_1, is a geometrically realized eigenvector of $\bar{R}_u$ in $u^\perp \cap \xi_1^\perp$ corresponding to the eigenvalue c_2.*

b) There exist $c_1, c_2 \in \mathbb{R}$ such that, for any $x, y, v \in T_pM$,

$$R(x,y)v = S^*(x,y)v - S_*(x,y)v + c_2 R_I^0(x,y)v + \frac{c_1 - c_2}{3} R^\varphi(x,y)v. \tag{16}$$

c) 1. *For any $v \in span\{\xi_1\}$, $x \in \xi_1^\perp$ we have*

$$R(x,v)v = (\eta^1(v))^2 (x - \widetilde{\eta}(x)\xi_2).$$

2. *There exist $c_1, c_2 \in \mathbb{R}$ such that, for any $v, y, x \in \xi_1^\perp$, we have*

$$\begin{aligned} R(x,y)v = {} & \eta^2(v)\left(\eta^2(y)x - \eta^2(x)y\right) + \left(g(y,v)\eta^2(x) - g(x,v)\eta^2(y)\right)\widetilde{\xi} \\ & + g(\varphi y, \varphi v)\varphi^2 x - g(\varphi x, \varphi v)\varphi^2 y + c_2 R_I^0(x,y)v + \frac{c_1 - c_2}{3} R^\varphi(x,y)v. \end{aligned}$$

Furthermore, if one of the above statements is verified, then the eigenvalues c_1 and c_2 satisfy $c_1 - 4c_2 + 3 = 0$.

Proof. We begin proving $a) \Rightarrow b)$. For this purpose, we consider the curvature-like map F on T_pM given by

$$F(x,y,v,z) = R(x,y,v,z) + \mu g(R_I^0(x,y)v,z) + \tau g(R^\varphi(x,y)v,z), \tag{17}$$

where $\mu, \tau \in \mathbb{R}$.

We want to apply Lemma 5.3 to F. About the hypotheses of Lemma 5.3, we see at once that F satisfies (11) since $F = R$ if one of its four arguments is a characteristic vector and (2) hold. Thus we must only compute $F(u,y,u,y)$, for any degenerate vector $u \in N_\varphi(\xi_1)$ and $y \in u^\perp \cap \operatorname{Im}\varphi$.

Namely, considering a null vector $u \in N_\varphi(\xi_1)$ and a vector $y \in u^\perp \cap \operatorname{Im}\varphi$, we find the suitable values of μ and τ in $\mathbb{R}$ for which F vanishes on degenerate 2-plane $\pi = span\{u, y\}$.

Let $y_1 = \varphi x_1 \in u^\perp \cap \xi_1^\perp$, then

$$\begin{aligned} F(y_1,u,u,y_1) &= -g(R(y_1,u)u,y_1) + \mu g(R_I^0(y_1,u)u,y_1) + \tau g(R^\varphi(y_1,u)u,y_1) \\ &= -c_1 + \mu + 3\tau. \end{aligned} \tag{18}$$

Analogously, if $y_2, y_2' \in u^\perp \cap \xi_1^\perp$ are geometrically realized orthonormal eigenvectors of $\bar{R}_u$ with respect to the eigenvalue c_2, then we have

$$\begin{aligned} F(y_2,u,u,y_2) &= -g(R(y_2,u)u,y_2) + \mu g(R_I^0(y_2,u)u,y_2) + \tau g(R^\varphi(y_2,u)u,y_2)) \\ &= (-c_2 + \mu), \end{aligned} \tag{19}$$

$$F(y_2,u,u,y_2') = -g(R(y_2,u)u,y_2') + \mu g(R_I^0(y_2,u)u,y_2') + \tau g(R^\varphi(y_2,u)u,y_2')) = 0, \tag{20}$$

$$F(y_2,u,u,y_1) = -g(R(y_2,u)u,y_1) + \mu g(R_I^0(y_2,u)u,y_1) + \tau g(R^\varphi(y_2,u)u,y_1)) = 0. \tag{21}$$

Now, imposing $F = 0$, we get

$$\mu = c_2 \text{ and } \tau = \frac{c_1 - c_2}{3}. \tag{22}$$

So, since a vector y of $u^\perp \cap \operatorname{Im}\varphi$ can be written as $y = ay_1 + b_j y_2^j$, where y_1 and y_2^j are geometrically realized eigenvectors of $\bar{R}_u$ in $u^\perp \cap \xi_1^\perp$ corresponding to c_1 and c_2, respectively. By (18), (19), (20) and (21) we have

$$\begin{aligned}F(y,u,u,y) &= a^2 F(y_1,u,u,y_1) + ab_j F(y_1,u,u,y_2^j) + ab_k F(y_2^k,u,u,y_1)\\ &\quad + b_k b_j F(y_2^k,u,u,y_2^j) = 0.\end{aligned}$$

Therefore, applying Lemma 5.3, we obtain $F(x,y,v,z) = g(S_*(x,y)v,z) - g(S^*(x,y)v,z)$, for any $x,y,v,z \in T_pM$. Then, by (17) and (22), we get

$$R(x,y,v,z) = g(S_*(x,y)v,z) - g(S^*(x,y)v,z) - c_2 g(R_I^0(x,y)v,z) - \frac{c_1-c_2}{3} g(R^\varphi(x,y)v,z).$$

Thus one obtains

$$R(x,y)v = -S_*(x,y)v + S^*(x,y)v + c_2 R_I^0(x,y)v + \frac{c_1-c_2}{3} R^\varphi(x,y)v.$$

The proof *b)* $\Rightarrow$ *c)* is straightforward. In fact, for any $v \in span\{\xi_1\}$, $x \in \xi_1^\perp$, we have

$$R(x,v)v = S^*(x,v)v = (\eta^1(v))^2(x + \widetilde{\eta}(x)\xi_1 + \varepsilon_1\widetilde{\eta}(x)\widetilde{\xi}) = (\eta^1(v))^2\,(x - \widetilde{\eta}(x)\xi_2)\,,$$

so obtaining *c)1.*

For any $v,y,x \in \xi_1^\perp$, by *b)* one gets

$$\begin{aligned}R(x,y)v &= \eta^2(y)\eta^2(v)x - \eta^2(x)\eta^2(v)y + (g(y,v)\eta^2(x) - g(x,v)\eta^2(y))\widetilde{\xi}\\ &\quad + (-S_* + c_2 R_I^0 + \frac{c_1-c_2}{3} R^\varphi)(x,y)v,\end{aligned}$$

that is *c)2.*

Finally, we prove *c)* $\Rightarrow$ *a)*. Consider $u \in N(\xi_1)$, $u = \xi_1 + x_1$ and put $y_1 = \varphi x_1$. One has

$$R(y_1,u)u = R(y_1,\xi_1))\xi_1 + R(y_1,x_1))\xi_1 + R(y_1,\xi_1))x_1 + R(y_1,x_1))x_1. \tag{23}$$

So, using *c)1.* and *c)2.*, we have

$$R(y_1,\xi_1)\xi_1 = y_1 \quad \text{and} \quad R(y_1,x_1)x_1 = (c_1-1)y_1.$$

By *c)2.*, for any $v \in \xi_1^\perp$, it is clear that

$$g(R(y_1,x_1)\xi_1,v) = -g(R(y_1,x_1)v,\xi_1) = 0, \quad g(R(y_1,\xi_1)x_1,v) = g(R(x_1,v)y_1,\xi_1) = 0.$$

On the other hand, if $v = \xi_1$, then

$$g(R(y_1,x_1)\xi_1,\xi_1) = 0, \quad g(R(y_1,\xi_1)x_1,\xi_1) = -g(y_1,x_1) = 0.$$

Hence, $\bar{R}_u(\overline{y}_1) = c_1\overline{y}_1$.

Analogously, considering $y_2 \in (span\{x_1,y_1\})^\perp \cap \operatorname{Im}\varphi$, then

$$R(y_2,u)u = R(y_2,\xi_1))\xi_1 + R(y_2,x_1))\xi_1 + R(y_2,\xi_1))x_1 + R(y_2,x_1))x_1.$$

As for y_1, using *c)*, it is easy to check that $R(x_1,v)y_2 = 0$ and $R(y_2,x_1)v = 0$. Moreover, applying *c)1.*, we get

$$R(y_2,\xi_1)\xi_1 = y_2.$$

The relation *c)2.* implies

$$R(y_2, x_1)x_1 = (c_2 - 1)y_2.$$

Therefore we have $\bar{R}_u(\overline{y}_2) = c_2\overline{y}_2$.

Finally, to prove the φ-null Osserman condition, we have to check that every eigenvalue does not depend on $u \in N_\varphi(\xi_1)$. In fact, by *c)* we find

$$\begin{aligned}R(\xi_2, \xi_1)\xi_1 &= 0,\\ R(\xi_2, x_1)x_1 &= g(x_1, x_1)\widetilde{\xi} = \xi_1 + \xi_2.\end{aligned}$$

It is easy to see that, for any $v \in \xi_1^\perp$

$$\begin{aligned}g(R(\xi_2, \xi_1)x_1, v) + g(R(\xi_2, x_1)\xi_1, v) &= -2g(R(\xi_2, x_1)v, \xi_1) + g(R(\xi_2, v)x_1, \xi_1)\\ &= 2g(x_1, v) - g(x_1, v) = g(x_1, v).\end{aligned}$$

Moreover, since

$$g(R(\xi_2, \xi_1)x_1, \xi_1) + g(R(\xi_2, x_1)\xi_1, \xi_1) = -g(R(\xi_2, \xi_1)\xi_1, x_1) = 0,$$

one obtains $R(\xi_2, \xi_1)x_1 + R(\xi_2, x_1)\xi_1 = x_1$. Then one gets $R(\xi_2, u)u = \xi_2 + \xi_1 + x_1 = \xi_2 + u$, so $\bar{R}_u(\overline{\xi}_2) = \overline{\xi}_2$. This proves *a)1.* and *a)2.*.

Finally, suppose M satisfies *a)* or *b)* or *c)*, by (2) and by *b)*, since $\varepsilon = 0$, for any $x, y, v, z \in \operatorname{Im}\varphi$ one gets

$$\begin{aligned}R(x, y, \varphi v, z) + R(x, y, v, \varphi z) &= (1 - c_2)\left(g(R_I^0(x, y)\varphi v, z) + g(R_I^0(x, y)v, \varphi z)\right)\\ &- \frac{c_1 - c_2}{3}\left(g(R^\varphi(x, y)\varphi v, z) + g(R^\varphi(x, y)v, \varphi z)\right) = (-1 + c_2 - \frac{c_1 - c_2}{3})P(x, y; v, z) = 0,\end{aligned}$$

which implies $c_1 - 4c_2 + 3 = 0$. This concludes the proof. □

An important and immediate consequence of the above theorem and Theorem 4.4 is the following corollary. In fact it gives a characterization of a particular subclass of φ-null Osserman Lorentz $\mathcal{S}$-manifolds.

Corollary 5.5. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, 2\}$, $n > 1$, be a $(2n+2)$-dimensional Lorentz $\mathcal{S}$-manifold with timelike vector field ξ_1. Then M satisfies one of the statements of Theorem 5.4 if and only if it is a Lorentz $\mathcal{S}$-space form with φ-sectional curvature $c = c_1 - 1 = 4c_2 - 4$.*

Proof. Let M be φ-null Osserman with respect to ξ_1 and, for $u \in N_\varphi(\xi_1)$, let c_1 and c_2 be the eigenvalues of $\bar{R}_u$, corresponding to eigenvectors in $\operatorname{Im}\varphi$. By the above theorem, we know that the curvature tensor field R is given by the statement *b)* in the above theorem. Therefore, if x is a non lightlike vector of $\operatorname{Im}\varphi$, then

$$R(x, \varphi x, x, \varphi x) = -(1 - c_2)g(x, x)^2 - \frac{c_1 - c_2}{3}(-1 - 2)g(x, x)^2 = (c_1 - 1)g(x, x)^2.$$

Hence the φ-sectional curvature c is constant and $c = c_1 - 1$. Since $c_1 - 4c_2 + 3 = 0$, then $c = 4c_2 - 4$.

The converse statement follows by Theorem 4.4. □

Analogously, if the Jacobi operator has a single eigenvalue corresponding to eigenvectors in $\operatorname{Im}\varphi$, we can prove the following corollary.

Corollary 5.6. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1,2\}$, $n > 1$ be a $(2n+2)$-dimensional Lorentz $\mathcal{S}$-manifold with timelike vector field ξ_1. Then M satisfies one of the statements of Theorem 5.4, with $c_1 = c_2 = \lambda$, if and only if it is a Lorentz $\mathcal{S}$-space form with φ-sectional curvature $c = 0$. Moreover, $\lambda = 1$.*

Proof. If M satisfies one of the statements of Theorem 5.4 with $c_1 = c_2 = \lambda$, by Corollary 5.5, one has that M is a Lorentz $\mathcal{S}$-space form with φ-sectional $c = \lambda - 1 = 4\lambda - 4$, which implies $c = 0$ and $\lambda = 1$.

Vice versa, assume that M is a Lorentz $\mathcal{S}$-space form with $c = 0$. By Corollary 5.5, M verifies one of the statements of Theorem 5.4 with $c_1 - 1 = 0 = 4(c_2 - 1)$, which implies $\lambda = c_1 = c_2 = 1$. □

Now we deal with the case $n = 1$. It is a special case because, for $u \in N_\varphi(\xi_1)$, the only eigenvector of the Jacobi operator $\bar{R}_u$ in $\operatorname{Im}\varphi$ is realized geometrically by the vector φx_1 in $u^\perp \cap \xi_1^\perp$. In this case we have a non compact example. It is carried out by $\mathbb{R}^4$ endowed with the Lorentz $\mathcal{S}$-structure, constructed as follows ([5]). Denoting the standard coordinates with $\{x, y, z^1, z^2\}$, we define on $\mathbb{R}^4$ two vector fields and two 1-forms putting

$$\xi_\alpha = \frac{\partial}{\partial z^\alpha}, \quad \eta^\alpha = dz^\alpha + y dx,$$

for any $\alpha \in \{1,2\}$. The tensor fields φ and g are given in the standard basis by

$$F := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & y & 0 & 0 \end{pmatrix} \qquad G := \begin{pmatrix} \frac{1}{2} & 0 & -y & y \\ 0 & \frac{1}{2} & 0 & 0 \\ -y & 0 & -1 & 0 \\ y & 0 & 0 & 1 \end{pmatrix}$$

respectively. It is easy to check that $(\mathbb{R}^4, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1,2\}$, is a Lorentz $\mathcal{S}$-manifold with different causal type of the characteristic vector fields. Moreover it is a Lorentz space form with φ-sectional curvature $c = 0$. Therefore, by (1), one obtains

$$\begin{aligned} R(X,Y,V) = {} & \widetilde{\eta}(X) g(\varphi V, \varphi Y) \sum_{\alpha=1}^{2} \xi_\alpha - \widetilde{\eta}(Y) g(\varphi V, \varphi X) \sum_{\alpha=1}^{2} \xi_\alpha - \widetilde{\eta}(Y)\widetilde{\eta}(V)\varphi^2 X \\ & + \widetilde{\eta}(V)\widetilde{\eta}(X)\varphi^2 Y, \end{aligned}$$

for any $X, Y, V \in \mathfrak{X}(\mathbb{R}^4)$. Since $\operatorname{Im}\varphi = \langle X, Y \rangle$ where $X = \sqrt{2}(\frac{\partial}{\partial x} - y\xi_1 - y\xi_2)$ and $Y = \sqrt{2}\frac{\partial}{\partial y}$, one has

$$\bar{R}_u \varphi Z = \varphi Z, \quad \bar{R}_u \xi_2 = \xi_2,$$

for any $Z = aX + bY$ and $u = \xi_1 + Z$ where $a^2 + b^2 = 1$. Then the only eigenvalue of $\bar{R}_u$, $u \in N_\varphi(\xi_1)$, is 1.

Using arguments similar to Theorem 5.4, we get a general result about 4-dimensional φ-null Osserman manifolds.

Proposition 5.7. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1,2\}$, be a 4-dimensional Lorentz $\mathcal{S}$ manifold with a timelike characteristic vector field ξ_1. Then M is φ-null Osserman with respect to ξ_1 and φx_1 realizes geometrically the eigenvector of $\bar{R}_u$ corresponding to c_1 in $u^\perp \cap \xi_1^\perp$ if and only if M is a Lorentz $\mathcal{S}$-space form with φ-sectional curvature $c_1 - 1$.*

Proof. Arguing as in the proof of Theorem 5.4, we have the following equation

$$3\tau + \mu = c_1$$

Therefore one deduces

$$R(x,y)v = S^*(x,y)v - S_*(x,y)v + \mu R_I^0(x,y)v + \frac{c_1-\mu}{3} R^{\varphi}(x,y)v.$$

Combining the above equation with the last condition in (2), it follows $\mu = \frac{3+c_1}{4}$. Thus one has

$$R(x,y)v = S^*(x,y)v - S_*(x,y)v + \frac{3+c_1}{4} R_I^0(x,y)v + \frac{c_1-1}{4} R^{\varphi}(x,y)v.$$

Now, for any unit vector $x \in \operatorname{Im}\varphi$, one has

$$H(x) = -1 + \frac{3+c_1}{4} + \frac{3c_1-3}{4} = c_1 - 1.$$

$\square$

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